

The Roche problem: some analytics

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ABSTRACT

Exact analytical formulae are derived for the potential and mass ratio as a function of Lagrangian points position, in the classical Roche model of the close binary stars.

Subject headings: stars: rotation—binaries: close

1. Introduction

The Roche model is widely used in interpretation of the close binary star observations. Several authors derived the various approximations to solve the Roche problem and presented numerical tables, see e.g. Plavec & Kratochvil (1964); Kippenhahn & Thomas (1970); Eggleton (1983); Mochnacki (1984); Morris (1985, 1994). Not intending to solve the whole problem analytically, I rather show here the way how some analytical relations can be found. The idea is to reverse the problem: instead of finding, e.g., the first Lagrangian point, x_1 , as function of binary mass ratio q we seek solution for q as function of x_1 .

2. Basic equation

The basic equation of the classic Roche problem is the formula for surfaces of the (primary) star as equipotential :

$$\Psi(x, y, z) = \left(x - \frac{q}{1+q}\right)^2 + y^2 + \frac{2q}{(1+q) \sqrt{(-1+x)^2 + y^2 + z^2}} + \frac{2}{(1+q) \sqrt{x^2 + y^2 + z^2}}. \quad (1)$$

I use in eq. (1) notations of Mochnacki (1984): the x axis is aligned along the stars' centers, z axis is parallel to the rotation axis, Ψ is "normalized" potential, in units of $G(M_1 + M_2)/(2A)$, $q = M_2/M_1 < 1$, and x, y, z are in units of A , the distance between the centers of binary components with masses M_1 and M_2 . As I here consider only the first (inner) and second (outer) Lagrangian points problem, it is sufficient to consider the eq. (1) at the x axis.

3. The first Lagrangian point

We write down eq. (1) with $y = z = 0$, $0 < x < 1$, and look for the minimum of the function:

$$\Psi_1(x) = \frac{2q}{(1+q)(1-x)} + \frac{2}{(1+q)x} + \left(x - \frac{q}{1+q}\right)^2. \quad (2)$$

at some $x = x_1$, with $0 < x_1 < 1$. The important observation, from eq. (2), is that $\Psi_1(q, x) = \Psi_1(1/q, 1 - x)$, if $0 < x < 1$ (not in general case!). From eq. (2), we find the condition $d\Psi_1(x)/dx = 0$, which we rewrite after some algebra as function of $q(x_1)$:

$$q(x_1) = \frac{(1 - x_1)^3 (1 + x_1 + x_1^2)}{x_1^3 (3 - 3x_1 + x_1^2)}. \quad (3)$$

We notice, from eq. (3), the elegant relation (also having a clear physical meaning) $q(x_1)q(1 - x_1) = 1$. We underline that eq. (3) gives the fully analytical and *exact* relation between values of q and of the first Lagrangian point x_1 . See, e.g. table 1 in (Mochnacki 1984), where Q and $X1$ stand for our q and x_1 , respectively, and compare the difference in difficulty of calculations by method of Mochnacki (1984) and by formulas (3,4,6,9) of this note.

Now in order to find value of the potential corresponding to the first Lagrangian point, we may use the eqs. (2) and (3) together, or, in the spirit of this note, exclude q from eqs. (2) and (3) and find explicit function $\Psi_1(x_1)$:

$$\Psi_1(x_1) = \frac{3 - 12t + 15t^2 - 10t^3 - 4t^4}{(-1 + 2t + t^2)^2}; \quad t = x_1(1 - x_1). \quad (4)$$

We introduced the additional variable t into Eq. (4) in order to explicitly show that $\Psi_1(x_1) = \Psi_1(1 - x_1)$ if $0 < x_1 < 1$ (not in general case!). Eq. (4) gives the fully analytical and *exact* relation between the values of the first Lagrangian point x_1 and of the corresponding potential $\Psi_1(x_1)$. See, e.g. table 1 in (Mochnacki 1984), where $C1$ and $X1$ stand for our $\Psi_1(x_1)$ and x_1 , respectively.

4. The second Lagrangian point

Now we look for minimum of the function (note the difference from eq. [2]):

$$\Psi_2(x) = \frac{2q}{(1+q)(x-1)} + \frac{2}{(1+q)x} + \left(x - \frac{q}{1+q}\right)^2, \quad (5)$$

at some $x = x_2$, with $1 < x_2 < 2$. Repeating the procedure of the section 3, we get the solution for q as function of x_2 :

$$q(x_2) = \frac{(x_2 - 1)^3 (1 + x_2 + x_2^2)}{x_2^2 (2 - x_2) (1 - x_2 + x_2^2)}. \quad (6)$$

We notice that by contrast with eq. (3), it is not evident from eq. (6) that we have the relation $q(x_2)q(1 - x_2) = 1$, because here $x_2 > 1$, but $(1 - x_2) < 0$, while the eq. (6) is derived under condition $1 < x_2 < 2$; in the case of eq. (3), both x_1 and $(1 - x_1)$ were in the (open) interval $(0,1)$.

To prove the validity of the relation $q(x_2)q(1 - x_2) = 1$, we should return to the basic equation (1), put there $y = z = 0, x < 0$, and look for the minimum of the function:

$$\Psi_3(x) = \frac{2q}{(1+q)(1-x)} - \frac{2}{(1+q)x} + \left(x - \frac{q}{1+q}\right)^2, \quad (7)$$

at some $x = x_3$, with $x_3 < 0$. Note the differences between functions $\Psi_1(x)$, $\Psi_2(x)$, and $\Psi_3(x)$. Repeating above procedure for $\Psi_3(x)$, we get:

$$q(x_3) = \frac{(2 - x_3) x_3^2 (1 - x_3 + x_3^2)}{(x_3 - 1)^3 (1 + x_3 + x_3^2)}. \quad (8)$$

Now, from eqs. (6) and (8), we have $q(x_2)q(x_3) = 1$, if $x_3 = 1 - x_2$, and if $x_2 > 1$, $x_3 = 1 - x_2 < 0$, QED. Eq. (6) (together with eq. [8]) gives the fully analytical and *exact* relation between q and the value of the second Lagrangian point x_2 . See, e.g. table 1 in (Mochnacki 1984), where Q and $X2$ stand for our q and x_2 , respectively.

Now in order to find value of the potential corresponding to the second Lagrangian point, we may use the eqs. (5) and (6) together, or, again in the spirit of this note, exclude q from eqs. (5) and (6) and find explicit function $\Psi_2(x_2)$:

$$\Psi_2(x_2) = \frac{-1 - 4x_2 + 27x_2^2 - 36x_2^3 + 9x_2^4 + 18x_2^5 - 14x_2^6 + 4x_2^7}{(-1 + 2x_2 + x_2^2 - 2x_2^3 + x_2^4)^2}. \quad (9)$$

Eq. (9) gives the fully analytical and *exact* relation between the values of the second Lagrangian point x_2 and of the corresponding potential $\Psi_2(x_2)$. See, e.g. table 1 in (Mochnacki 1984), where $C2$ and $X2$ stand for our $\Psi_2(x_2)$ and x_2 , respectively.

5. Summary

In this short note we present the exact analytical relations for the first and second Lagrangian point of the classical Roche problem. In practice, the "exact" formulas are not necessarily the most convenient ones, hence different approximate expressions in literature (see (Kopal 1959; Plavec & Kratochvil 1964; Kippenhahn & Thomas 1970; Eggleton 1983; Mochnacki 1984; Morris 1985, 1994), among others), and some relevant approximate formulas will be given elsewhere. Still, the exact formulas have their own beauty and are more relevant as solutions to the classical problems such as the problem by Roche (1847).

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